Problem 12) The smallest n for which the problem is meaningful is $n = 2$. In this case the product of the lengths $x_1 x_2 = x_1(L - x_1)$ is readily maximized by setting the derivative with respect to x_1 equal to zero. We will have

$$\frac{\mathrm{d}}{\mathrm{d}x_1}[x_1(L - x_1)] = L - 2x_1 = 0 \quad \rightarrow \quad x_1 = L/2 \quad \rightarrow \quad x_2 = L - x_1 = L/2.$$

Assume the product is known to be a maximum for some $n \geq 2$ when $x_1 = x_2 = \cdots = x_n = L/n$. What happens if we decide to divide the stick into $n + 1$ pieces? Fix the length of the first piece at x_1. By assumption, the product $x_1 x_2 \cdots x_n\, x_{n+1}$ will then be maximized if $x_2 = x_3 = \cdots = x_{n+1} = (L - x_1)/n$. Therefore, we must choose x_1 such that $x_1 x_2 \cdots x_n x_{n+1} = x_1[(L - x_1)/n]^n$ is a maximum. Differentiation with respect to x_1 and setting the derivative equal to zero now yields

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left[x_1\left(\frac{L - x_1}{n}\right)^n\right] = \left(\frac{L - x_1}{n}\right)^n + x_1 n\,(-1/n)\left(\frac{L - x_1}{n}\right)^{n-1}$$

$$= \left(\frac{L - x_1}{n}\right)^{n-1}\left[\frac{L - (1+n)x_1}{n}\right] = 0 \quad \rightarrow \quad \begin{cases} x_1 = L; \\ x_1 = \dfrac{L}{n+1}. \end{cases}$$

The first solution, $x_1 = L$, is unacceptable as it leads to the product $x_1 x_2 \cdots x_n x_{n+1} = 0$. The second solution, $x_1 = L/(n + 1)$, however, shows that the maximum product is obtained when all $n + 1$ segments have equal lengths, i.e., $L/(n + 1)$. The proof by induction is now complete.